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VECTOR BUNDLES TRIVIALIZED BY PROPER MORPHISMS AND THE FUNDAMENTAL GROUP SCHEME, II

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Abstract. Let X be a projective and smooth variety over an algebraically closed field k. Let $f : Y \longrightarrow X$ be a proper and surjective morphism of k–varieties. Assuming that f is separable, we prove that the Tannakian category associated to the vector bundles E on X such that f^*E is trivial is equivalent to the category of representations of a finite and etale group scheme. We give a counterexample to this conclusion in the absence of separability.

1. Introduction

The present work is a continuation of [BdS10], giving some applications of the main result in [BdS10] which throw light on the nature of the fundamental group scheme of Nori [No76] for a smooth projective variety.

Let X be a smooth projective variety over an algebraically closed field k. The fundamental group scheme of X is the affine group scheme obtained from the (Tannakian) category of essentially finite vector bundles on X (see Definition 3). The main theorem of [BdS10] says that a vector bundle E over X is essentially finite if and only if there is a proper k–scheme Y and a surjective morphism $f : Y \longrightarrow X$ such that f^*E is trivial. As an application of this theorem, we prove the following :

Theorem 1. *Let X be a smooth and projective variety over the algebraically closed field k, $x_0 : \mathrm{Spec}(k) \longrightarrow X$ a point, and $f : Y \longrightarrow X$ a proper and surjective morphism of varieties.*

(i) *The full subcategory of $\mathbf{VB}(X)$*

$$\mathcal{T}_Y(X) = \{V \in \mathbf{VB}(X) : \ f^*V \ \textit{is trivial}\}$$

is Tannakian. The functor $x_0^* : \mathcal{T}_Y(X) \longrightarrow$ (k–mod) is a fibre functor.

(ii) *Assume that f is separable. Let $G(Y/X)$ denote the affine group scheme obtained from $\mathcal{T}_Y(X)$ and x_0^*. Then $G(Y/X)$ is finite and etale.*

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(iii) *If the separability assumption on f is removed, then there exists a counterexample to the conclusion in* (ii) *in which $G(Y/X)$ is not a finite group scheme.*

Part (i) of the above theorem is routine, see Lemma 7. Part (ii) is the subject of Theorem 8, while the counterexample alluded to in (iii) is produced in section 4.1.

2. Preliminaries

2.1. **Notation and terminology.** Throughout k will stand for an algebraically closed field; by a variety we mean an integral scheme of finite type over k.

Let V be a normal variety. Its field of rational functions will be denoted by $R(V)$. We will let $\mathrm{Val}(V)$ denote the set of discrete valuations of $R(V)$ associated to V: a discrete valuation $v : R(V) \longrightarrow \mathbb{Z} \cup \{\infty\}$ belongs to $\mathrm{Val}(V)$ if and only if there exists a point ξ of codimension one in V such that $\mathcal{O}_{V,\xi} = \{\varphi \in R(V) : v(\varphi) \geq 0\}$.

Given a finite extension of fields L/K and a set of discrete valuations S of K, we say that L is unramified above S if for each discrete valuation v of S and each prolongation w of v to L, the ramification index $e(w/v) = 1$ and the extension of residue fields is separable.

A dominant morphism $f : W \longrightarrow V$ between two varieties is *separable* if the extension of function fields $R(W)/R(V)$ is *separable*. (This differs from the homonymous notion defined in [SGA1 X, Definition 1.1].)

A vector bundle over a scheme is a locally free coherent sheaf. The category of all vector bundles on X will be denoted by $\mathbf{VB}(X)$. If E is a vector bundle over the k–scheme X, we will say that E comes from a representation of the étale fundamental group if there exists a finite group Γ, a representation

$$\rho : \Gamma \longrightarrow \mathrm{GL}_m(k)$$

and an étale Galois covering of group Γ, $Y \longrightarrow X$, such that

$$E \cong Y \times^{\Gamma} k^{\oplus m}.$$

(For the general definition of the contracted product of a torsor and a representation, see e.g. [J87, I 5.8, 5.14].)

Given an affine group scheme G over k, we will let $\mathrm{Rep}(G)$ denote the category of all finite dimensional representations of G [Wa79, Ch. 3]. A morphism of affine group schemes $f : G \longrightarrow H$ is a quotient morphism if it is faithfully flat, or, equivalently, if the homomorphism induced on the function rings is injective [Wa79, Ch. 14].

If $\mathcal{T}$ is a Tannakian category over k [DM82] and $V \in \mathcal{T}$, we define the monodromy category of V to be the smallest Tannakian sub–category of $\mathcal{T}$ containing V: it will be denoted by $\langle V; \mathcal{T}\rangle_\otimes$.

Consider the neutral Tannakian category $\mathrm{Rep}(G)$ over k. For any $V \in \mathrm{Rep}(G)$, the category $\langle V; \mathrm{Rep}(G)\rangle_\otimes$ is equivalent to the category of representations of the image of the tautological homomorphism $\rho_V : G \longrightarrow \mathrm{GL}(V)$. This image will be called the monodromy group of V; see Definition 2.5 and the remark after it in [BdS10] for more information.

2.2. **Vector bundles trivialized by proper and surjective morphisms.** Let X be a smooth and projective variety over k (recall that k is algebraically closed).

Definition 2 (Property T)**.** A vector bundle E over X is said to have *property (T)* if there exists a proper k–scheme Y together with a surjective (proper) morphism $f : Y \longrightarrow X$ such that the pull–back f^*E is trivial.

The main result of [BdS10] relates property (T) to the more sophisticated notion of essential finiteness.

Definition 3. Following Nori [No76], we say that a vector bundle over X is *essentially finite* if there exists a finite group scheme G, a G–torsor $P \longrightarrow X$ and a representation $\rho : G \longrightarrow \mathrm{GL}(V)$, such that

$$P \times^G V \cong E .$$

The category of all essentially finite vector bundles over X will be denoted by $\mathbf{EF}(X)$.

Remark 4. Every essentially finite vector bundle enjoys property (T) as these are trivialized by a torsor under a finite group scheme.

The category $\mathbf{EF}(X)$ is Tannakian [No76]. The above definition of essential finiteness is not the one presented in [No76], but a consequence of the results of that work.

Theorem 5. [BdS10, Theorem 1.1] *A vector bundle E over X is essentially finite if and only if it satisfies property (T).*

The reader is urged to read Remark 14 at the end of this text to be directed to another proof of the case where $\dim X = 1$; this proof was suggested to us by Parameswaran and is based on [BP, §6] (which contains very interesting conceptual advancements). Also, we indicate that recently Antei and Mehta put forward a generalisation of Theorem 5 in the case where X is only normal [AM].

It should be clarified that the smoothness condition on Theorem 5 cannot be dropped; this is shown by the following example:

Example 6. Let $X \subset \mathbb{P}^2_k$ be the nodal cubic defined by $(y^2z = x^3 + x^2z)$. Let

$$f : \mathbb{P}^1 \longrightarrow X, \quad (s : t) \mapsto (s^2t - t^3 : s^3 - st^2 : t^3),$$

be the birational morphism which identifies the points $(1 : 1)$ and $(-1 : 1)$. It is well–known that $\mathrm{Pic}^0(X) = k^*$, so that any line bundle L of infinite order over X gives a counter–example to the generalization of Theorem 5 to the case where X is not normal.

2.3. **The fundamental group–scheme.** Fix a k–rational point $x_0 : \mathrm{Spec}(k) \longrightarrow X$. The essentially finite vector bundles with the fibre functor defined by sending any essentially finite vector bundle E to its fibre x_0^*E over x_0 form a neutral Tannakian category [DM82, Definition 2.19]. The corresponding affine group–scheme over k [DM82, Theorem 2.11] is called the *fundamental group–scheme* [No76], [No82]. This group–scheme will be denoted by $\Pi^{\mathrm{EF}}(X, x_0)$.

3. Vector bundles trivialized by separable proper morphisms

Throughout this section, we let X stand for a projective and smooth variety and $f : Y \longrightarrow X$ for a proper surjective morphism from a (proper) variety Y. We also choose a k–rational point $x_0 : \mathrm{Spec}(k) \longrightarrow X$.

3.1. **The object of our study.** For general terminology on Tannakian categories the reader should consult [DM82].

Lemma 7. *The full subcategory of* $\mathbf{VB}(X)$

$$\mathcal{T}_Y(X) = \{V \in \mathbf{VB}(X);\ f^*V \ \textit{is trivial}\}$$

is Tannakian. The functor $x_0^* : \mathcal{T}_Y(X) \longrightarrow$ *(k–mod) is a fibre functor.*

Proof. That $\mathcal{T}_Y(X)$ is stable by tensor products and direct sums is clear. That it is an abelian category is a consequence of the fact that all vector bundles in $\mathcal{T}_Y(X)$ are Nori–semistable, so that kernels and cokernels are always vector bundles; see [BdS10, Corollary 2.3] and [No76, Lemma 3.6]. Using this last remark, it is easy to understand why the functor x_0^* is exact and faithful. As $\mathcal{T}_Y(X)$ has only vector bundles as objects, the rigidity axiom for a Tannakian category is promptly satisfied. □

The affine group scheme obtained from $\mathcal{T}_Y(X)$ and the fibre functor x_0^* via the main Theorem of Tannakian categories [DM82, Theorem 2.11] will be denoted by $G(Y/X)$ in the sequel.

3.2. Finiteness of $G(Y/X)$ for separable morphisms.

Theorem 8. *We assume that $f : Y \longrightarrow X$ is separable.*

(1) If the vector bundle E is such that f^*E is trivial, then E is essentially finite and in fact comes from a representation of the étale fundamental group. Moreover, the monodromy group of E in the category $\mathbf{EF}(X)$ at the point $x_0 \in X(k)$ is a quotient of a fixed finite étale group scheme Γ^{nr}. (See Section 2.1 for definitions.)

(2) The group scheme $G(Y/X)$ is finite and etale.

The first step towards a proof of Theorem 8 (and also of [BdS10, Theorem 1.1]) is to consider the Stein factorization of f:

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[dr, "f"'] & Y' \arrow[d, "g"] \\ & X, \end{tikzcd}$$

where g is finite and $h_*(\mathcal{O}_Y) = \mathcal{O}_{Y'}$. The latter equality implies that the morphism $h^* : \mathbf{VB}(Y') \longrightarrow \mathbf{VB}(Y)$ is full and faithful, so that g^*E is already trivial.

Definition 9. Let $\varphi : V \longrightarrow X$ be a finite, surjective and separable morphism of varieties. By $R(V)^{\mathrm{nr}}$ we denote the maximal unramified intermediate extension of $R(V)/R(X)$, which is the compositum of all sub-extensions R of $R(V)/R(X)$ which are unramified over $\mathrm{Val}(V)$. We let

$$\varphi^{\mathrm{nr}} : V^{\mathrm{nr}} \longrightarrow X$$

denote the normalization of X in $R(V)^{\mathrm{nr}}$. If $R(V)/R(X)$ is Galois of group Γ, the Γ^{nr} denotes the Galois group of the extension $R(V)^{\mathrm{nr}}/R(X)$.

Proof of Theorem 8. (1): That E is essentially finite is the content of Theorem 5. For the remainder, it is enough to prove statement (1) in the theorem under the assumption that f is finite. There is also no loss of generality in assuming that the field extension $R(Y)/R(X)$ is Galois; let Γ be its Galois group.

We first prove that if Γ^{nr} is trivial, i.e. $f^{\mathrm{nr}} = \mathrm{id}_X$, then E is likewise. Let G be the finite group scheme associated, by Tannakian duality, to the category $\langle E; \mathbf{EF}(X)\rangle_\otimes$ via the point $x_0 \in X(k)$ (see Section 2.1). Let P be the G–torsor associated to E [No76, §2]; the functor

$$P \times^G (\bullet) \,:\, \mathrm{Rep}(G) \longrightarrow \langle E; \mathbf{EF}(X)\rangle_\otimes$$

induces an equivalence of monoidal categories. We denote by G^{et} the finite étale group scheme of connected components of G [Wa79, Chapter 6]. As P is connected [No82,

Proposition 3, p. 87], so is

$$P^{\mathrm{et}} := P/\ker\left(G \longrightarrow G^{\mathrm{et}}\right) = P \times^G G^{\mathrm{et}} .$$

Since $P^{\mathrm{et}} \longrightarrow X$ is an étale morphism, it follows that P^{et} is a normal *variety*.

Claim A: The triviality of f^*E implies the triviality of the G–torsor

$$P_Y := P \times_X Y \longrightarrow Y.$$

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a representation of G such that $E = P \times^G V$. It follows that ρ is a closed embedding and we are able to deduce the triviality of P_Y by using the triviality of $P_Y \times^G \mathrm{GL}(V)$ together with the fact that the natural map $H^1_{\mathrm{fppf}}(Y, G) \longrightarrow H^1_{\mathrm{fppf}}(Y, \mathrm{GL}(V))$ is injective (as the kernel is the set of all morphisms from Y to the affine scheme $\mathrm{GL}(V)/G$ [DG70, p. 373, III, §4, 4.6]).

Let $h : Y \longrightarrow P$ be the X–morphism derived from an isomorphism $P_Y \cong Y \times G$ and let $j : Y \longrightarrow P^{\mathrm{et}}$ be the morphism of X–schemes obtained from h. It is not hard to see that j takes the generic point of Y to the generic point of P^{et}, so j gives rise to a homomorphism of $R(X)$–fields $R(P^{\mathrm{et}}) \longrightarrow R(Y)$. Since $R(P^{\mathrm{et}})/R(X)$ is unramified above $\mathrm{Val}(X)$, we must have $R(P^{\mathrm{et}}) = R(X)$. As a consequence, $P^{\mathrm{et}} = X$ and thus G^{et} is trivial. This means that G is a local group scheme. We will now prove the following:

Claim B: If G is local, then the existence of an X–morphism $h : Y \longrightarrow P$ implies the triviality of P.

Let $\mathrm{Spec}(A) \subseteq X$ be an affine open and let $\mathrm{Spec}(B) \subseteq Y$ (respectively, $\mathrm{Spec}(S) \subseteq P$) be its pre–image in Y (respectively, in P). We then have a homomorphism of A–algebras $\eta : S \longrightarrow B$; let $S' \subseteq B$ be its image. Since

$$\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(A)$$

is a G–torsor, above any maximal ideal $\mathfrak{m} \subseteq A$, there exists only one maximal ideal of S; the same property is valid if we replace S by S'. Hence, the extension of fields defined by $S' \supseteq A$ must be *purely inseparable*. Because $R(Y)/R(X)$ is a separable extension, and A is a normal ring, it follows that $S' = A$. This allows one to construct a section $\sigma : X \longrightarrow P$. Therefore E is trivial. This proves Claim B.

Now we treat the general case. Since $f^{\mathrm{nr}} : Y^{\mathrm{nr}} \longrightarrow X$ is unramified above $\mathrm{Val}(X)$, the Zariski–Nagata purity Theorem (for the statement, see [SGA 1 X, 3.1]) permits us to conclude that f^{nr} is étale (in particular Y^{nr} a smooth projective variety over k). Moreover, $f^{\mathrm{nr}} : Y^{\mathrm{nr}} \longrightarrow X$ is an étale Galois covering of group Γ^{nr}. Let

$$g : Y \longrightarrow Y^{\mathrm{nr}}$$

denote the obvious morphism, we have $g^{\mathrm{nr}} = \mathrm{id}_{Y^{\mathrm{nr}}}$. Applying what was proved above to Y^{nr}, we conclude that $f^{\mathrm{nr}*}E$ is trivial. By [LS77, Proposition 1.2], we conclude that

$$E \cong Y^{\mathrm{nr}} \times^{\Gamma^{\mathrm{nr}}} V,$$

where V is a representation of Γ^{nr}. This proves that the monodromy group of E in $\mathbf{EF}(X)$ is a quotient of Γ^{nr}.

(2): The proof rests on the same sort of argument used for the proof of (1). As in (1), we assume that f is finite. Let

$$G(Y/X) := G = \varprojlim G_i$$

be the *profinite* group scheme associated to $\mathcal{T}_Y(X)$ via x_0^*; here each group G_i is finite and the transition morphisms $G_j \longrightarrow G_i$ are all faithfully flat. (The reader unfamiliar with this sort of structure argument will profit from [Wa79, 3.3] and [Wa79, 14.1].) Write $P \longrightarrow X$ for the universal G–torsor [No76, §2] and P_i for $P \times^G G_i$. We remark that Proposition 3 on p. 87 of [No82] proves that $\Gamma(P_i, \mathcal{O}_{P_i}) = k$. In this situation, we can find X–morphisms

$$h_i : Y \longrightarrow P_i\,.$$

(The details of the argument are given in the proof of (1) above.) Let G_i^{et} be the largest étale quotient of G_i [Wa79, Ch. 6]; the morphism

$$P_i^{\mathrm{et}} := P \times^G G_i^{\mathrm{et}} \longrightarrow X$$

is finite and étale and the number of k–rational points on a fiber equals $\operatorname{rank} G_i^{\mathrm{et}}$. From the surjectivity of the composition

$$Y \longrightarrow P_i \longrightarrow P_i^{\mathrm{et}},$$

the integers $\operatorname{rank} G_i^{\mathrm{et}}$ are bounded from above, so

$$G^{\mathrm{et}} := \varprojlim G_i^{\mathrm{et}} = G_{i_0}^{\mathrm{et}}$$

for some i_0. Let $X' := P_{i_0}^{\mathrm{et}} = P_i^{\mathrm{et}}$, it is a smooth and projective variety and the obvious morphism

$$P_i \longrightarrow P_i/G_i^0 = P_i/(\ker G_i \longrightarrow G_{i_0}^{\mathrm{et}}) = X', \quad i \geq i_0$$

gives $P_i \longrightarrow X'$ the structure of a torsor over X' under the structure group G_i^0. Moreover, since $\Gamma(P_i, \mathcal{O}_{P_i}) = k$, P_i cannot be trivial over X' unless $G_i^0 = \{e\}$. Employing the X'–morphisms $Y \longrightarrow P_i$, we see, using Claim B proved in part (1), that $G_i^0 = \{e\}$. This means that $G = G_{i_0}^{\mathrm{et}}$. □

4. Finiteness of $G(Y/X)$, reducedness of the universal torsor and base change properties

As in section 3, we let X stand for a projective and smooth variety and $f : Y \longrightarrow X$ for a proper surjective morphism from a (proper) variety Y. We also choose a k–rational point $x_0 : \mathrm{Spec}(k) \longrightarrow X$.

4.1. **An instance where $G(Y/X)$ is not finite and the universal torsor is not reduced.** Let $G(Y/X)$ be the affine fundamental group scheme associated to the Tannakian category

$$\mathcal{T}_Y(X)$$

by means of the fiber functor $x_0^* : \mathcal{T}_Y(X) \longrightarrow (k\text{–mod})$. If V is an object of $\mathcal{T}_Y(X)$ which is *stable* as a vector bundle (all vector bundles in $\mathcal{T}_Y(X)$ are semistable of slope zero [BdS10, Proposition 2.2]), the representation of $G(Y/X)$ obtained from V must be *irreducible*. Since a finite group scheme only has *finitely* many isomorphism classes of irreducible representations — these are all Jordan–Hölder components of the right regular representation [Wa79, 3.5] — we have a proved the following lemma.

Lemma 10. *If there are infinitely many non–isomorphic stable vector bundles in $\mathcal{T}_Y(X)$, then the group scheme $G(Y/X)$ is not finite.*

The existence of infinitely many stable bundles in $\mathcal{T}_Y(X)$ also causes the following particularity.

Proposition 11. *Assume that there are infinitely many non–isomorphic stable vector bundles in $\mathcal{T}_Y(X)$. Then there exists a finite quotient G_0 of $G(Y/X)$ and a G_0–torsor over X, call it P_0, such that*

(1) $\Gamma(P_0, \mathcal{O}_{P_0}) = k$ and

(2) the scheme P_0 is not reduced.

Moreover, in this case, the universal torsor $\widetilde{X} \longrightarrow X$ for the fundamental group scheme $\Pi^{\mathrm{EF}}(X, x_0)$ is not reduced as a scheme.

Proof. Let

$$G(Y/X) := G = \varprojlim G_i\,,$$

where each G_i is a finite group–scheme and the transition morphisms $G_j \longrightarrow G_i$ are faithfully flat, just as in the proof of Theorem 8. By Lemma 10 and the assumption, G is not a finite group scheme. We will show that the conclusion of the statement holds under the extra assumption that the group schemes G_i are all local. The general case can

be obtained from this one as in the proof of Theorem 8. Let $P \longrightarrow X$ be the universal G–torsor associated to $\mathcal{T}_Y(X) \subset \mathbf{EF}(X)$ via the constructions in [No76, §2]. The torsor P gives rise to G_i–torsors

$$\psi_i : P_i = P \times^G G_i \longrightarrow X.$$

Due to [No82, Proposition 3, p. 87], we have $\Gamma(P_i, \mathcal{O}_{P_i}) = k$. Since G_i is a local group scheme, for any field extension K/k, the map $\psi_i(K) : P_i(K) \longrightarrow X(K)$ is bijective, by [EGA I, 3.5.10, p. 116] ψ_i induces a bijection on the corresponding topological spaces. Hence, ψ_i is a homeomorphism and it follows that P_i is irreducible for each i. We assume that each P_i is also reduced. Proceeding as in the proof of Theorem 8 (see Claim A), there exists a X–morphism $h : Y \longrightarrow P_i$ for each i. This bounds $\deg \psi_i = \operatorname{rank} G_i$ by above and leads to a contradiction with the assumption that G is not finite.

The proof of the last statement is a direct consequence of what we just proved together with [No82, Proposition 3] and [EGA IV_3, 8.7.2]. $\square$

In view of Lemma 10 and Proposition 11, we can use [Pa07] to give an example of a smooth curve X having two extraordinary features: (1) there exists a finite morphism $Y \longrightarrow X$ such that $G(Y/X)$ is *not* finite and (2) the universal torsor $\widetilde{X}$ for the fundamental group scheme $\Pi^{\mathrm{EF}}(X, x_0)$ is *not* reduced. Indeed, let X be the smooth curve constructed in [Pa07, (3.1) and Proposition 4.1]: it is a smooth projective curve defined by a single explicit equation in $\mathbb{P}^2_k$; here k is any field of characteristic two. Let $f : Y \longrightarrow X$ be the fourth power of the Frobenius morphism (so Y is isomorphic to X a scheme). Pauly [Pa07, Proposition 4.1] constructs a locally free coherent sheaf over $X \times S$, where S is a positive dimensional k–scheme, such that for every $s \in S(k)$, the vector bundle $\mathcal{E}|X \times \{s\}$ is *stable* and $f^*(\mathcal{E}|X \times \{s\})$ is trivial. Furthermore, for two different points $s, t \in S(k)$, the sheaves $\mathcal{E}|X \times \{s\}$ and $\mathcal{E}|X \times \{t\}$ are not isomorphic. In other words, there are infinitely many isomorphism classes of stable vector bundles of fixed rank satisfying the condition that the pullback by f is trivial. By Lemma 10, the affine group scheme $G(Y/X)$ is not finite. From Proposition 11, it follows also that the universal torsor $\widetilde{X} \longrightarrow X$ is not reduced.

Remark 12. In [EHS08, Remark 2.4] the reader can find an example of an α_p–torsor over a reduced variety which is not reduced. The example we have just given shows that the situation can be bad even if the ambient variety is smooth.

4.2. **A link between the quantity of F–trivial vector bundles and the universal torsor.** We assume that k is of positive characteristic, and let $F : X \longrightarrow X$ be the absolute Frobenius morphism. Define

$$S(X, r, t) = \left\{ \begin{array}{l} \text{isomorphism classes of } \textit{stable} \text{ vector bundles of rank } r \\ \text{on } X, \text{ whose pull–back by } F^t \text{ is trivial} \end{array} \right\}$$

(Here we refrain from using the terminology F–trivial, since there is a question of stability which is not constant in the literature [Pa07], [MS08].) In their study of base change for the local fundamental group scheme and these bundles, Mehta and Subramanian [MS08] showed the following.

Theorem 13 ([MS08], Theorem, p. 208). *Let X be a smooth projective variety over k. The following are equivalent:*

(a) For any algebraically closed extension k'/k, any pair $r, t \in \mathbb{N}$ and any $E' \in S(X \otimes_k k'; r, t)$, there exists a vector bundle over X and an isomorphism $E \otimes_k k' \cong E'$.

(b) For any two given $r, t \in \mathbb{N}$, $S(X; r, t)$ is finite.

(c) The local fundamental group scheme of $X \otimes_k k'$ is obtained from the local fundamental group scheme of X by base change.

For the definition of the local fundamental group scheme, the reader should consult [MS08]. In Proposition 11 we have shown that

$$\left\{\begin{array}{c}\text{The universal torsor for the}\\ \text{fundamental group}\\ \text{scheme is a reduced scheme}\end{array}\right\} \Longrightarrow \{\text{Condition (b) in the above theorem holds.}\}$$

As Vikram Mehta made us realize, the reverse implication need not be true and the arguments to follow are due to him. To construct a counter–example, we consider an abelian threefold A and $\iota : X \hookrightarrow A$ a closed smooth surface defined by intersecting A with a hyperplane section of high degree in some projective embedding $A \hookrightarrow \mathbb{P}^N$. By "Lefschetz's Theorem" [BH, Theorem 1.1], we have an isomorphism

$$\Pi^{\mathrm{EF}}(\iota) : \Pi^{\mathrm{EF}}(X, x_0) \xrightarrow{\cong} \Pi^{\mathrm{EF}}(A, x_0)$$

so that, if $B \longrightarrow A$ is a pointed torsor under a finite group scheme with the property of being "Nori reduced" [No82, Proposition 3, p. 87], i.e.

$$H^0(B, \mathcal{O}_B) = k;$$

the same can then be said about the restriction of B to X. Using the torsors

$$[p] : A \longrightarrow A \quad ([p] \text{ is multiplication by } p),$$

we see that X admits a "Nori reduced" torsor under a finite group scheme which is not reduced *as a scheme*. (This follows from the factorization $[p] = VF$ and the fact that $F^{-1}(Z)$ is never reduced if $Z \subseteq A$ is a proper closed sub–scheme.) By another application of the "Lefschetz's Theorem" [BH, Theorem 1.1], we obtain a bijection

$$S(A, r, t) \xleftrightarrow{\sim} S(X, r, t).$$

Since the iteration of the Frobenius morphism $F_A^t : A \longrightarrow A$ sits in a commutative diagram

$$\begin{tikzcd} A \arrow[r, "{[p^t]}"] \arrow[dr] & A \\ & A, \arrow[u, "F_A"'] \end{tikzcd}$$

if ${F_A^t}^* E$ is trivial, then $[p^t]^* E$ is likewise; consequently, we obtain an injection

$$S(A, r, t) \hookrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of simple} \\ \text{representations of rank } r \text{ of } \ker [p] \end{array} \right\}.$$

This entails that $S(X, r, t)$ is always a finite set and we arrive at the desired counter–example to the above highlighted implication.

Remark 14 (Made after completion). In a recent discussion, Parameswaran called our attention to a simpler proof of the fact that a vector bundle E on X which becomes trivial after being pulled back by a finite morphism from a smooth and projective variety $f : Y \longrightarrow X$ in fact comes from a representation of the etale fundamental group of X (compare Theorem 8). The main idea is to use the *algebra*

$$f_*(\mathcal{O}_Y)_{\max}$$

associated to a separable and finite morphism $f : Y \longrightarrow X$ from a smooth projective variety Y to X (here the subscript "max" stands for the maximal semistable subsheaf). That this is in fact an algebra requires a proof and the reader is directed to [BP, Lemma 6.4]. One of the consequences of [BP] (which Parameswaran was kind enough to explain to the second author) is that $f_*(\mathcal{O}_Y)_{\max}$ is the maximal *etale* extension of $\mathcal{O}_X$ inside $f_*\mathcal{O}_Y$. Together with [BP, Proposition 6.8], the triviality of f^*E implies the triviality of the pull–back of E to the *finite etale* X–scheme $Y_{\max} = \operatorname{Spec} f_*(\mathcal{O}_Y)_{\max}$ and this enough to show that E is essentially finite. The reader should also note that in [BP, §6], the framework is such that the domain variety is smooth, which is not sufficient to obtain Theorem 8 directly; but it is possible that the methods in [BP] can be extended (for example, to a *normal domain variety*) to give another proof of Theorem 8.

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